Constrained optimization with second order conditions 2

Find, if they exist, the constrained extrema of the following function
$f(x, y) = 3x^2 + y^2 - x + 1$, subject to $x^2 + \frac{y^2}{4} = 1$

Solution

We want to find the extrema of the function:

$$f(x, y) = 3x^2 + y^2 - x + 1$$

subject to the constraint:

$$g(x, y) = x^2 + \frac{y^2}{4} - 1 = 0.$$

We use the method of Lagrange multipliers. We define the Lagrangian function:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \left(x^2 + \frac{y^2}{4} - 1 \right).$$

We calculate the partial derivatives of $\mathcal{L}$ with respect to x, y, and λ:

$$\frac{\partial \mathcal{L}}{\partial x} = 6x - 1 - 2\lambda x = 0,$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \frac{\lambda y}{2} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(x^2 + \frac{y^2}{4} - 1 \right) = 0.$$

Solving the system of equations:
1. From the equation $\frac{\partial \mathcal{L}}{\partial y} = 0$:

$$2y - \frac{\lambda y}{2} = 0 \implies y\left(2 - \frac{\lambda}{2}\right) = 0.$$

This implies two cases:
Case 1: $y = 0$.
Case 2: $2 - \frac{\lambda}{2} = 0 \implies \lambda = 4$.

Case 1: $y = 0$
With $y = 0$, from the constraint we get:

$$x^2 + \frac{0}{4} = 1 \implies x^2 = 1 \implies x = \pm 1.$$

Using the equation $\frac{\partial \mathcal{L}}{\partial x} = 0$:

$$6x - 1 - 2\lambda x = 0 \implies x(6 - 2\lambda) - 1 = 0.$$

For $x = 1$:

$$(6 - 2\lambda)(1) - 1 = 0 \implies 6 - 2\lambda - 1 = 0 \implies 2\lambda = 5 \implies \lambda = \frac{5}{2}.$$

For $x = -1$:

$$(6 - 2\lambda)(-1) - 1 = 0 \implies -6 + 2\lambda - 1 = 0 \implies 2\lambda = 7 \implies \lambda = \frac{7}{2}.$$

Case 2: $\lambda = 4$
With $\lambda = 4$, the equation $\frac{\partial \mathcal{L}}{\partial x} = 0$ becomes:

$$6x - 1 - 8x = 0 \implies -2x - 1 = 0 \implies x = -\frac{1}{2}.$$

Using the constraint:

$$\left(-\frac{1}{2}\right)^2 + \frac{y^2}{4} = 1 \implies \frac{1}{4} + \frac{y^2}{4} = 1 \implies y^2 = 3 \implies y = \pm\sqrt{3}.$$

Summary of critical points:

1. $(x, y) = (1, 0)$, $\lambda = \frac{5}{2}$.
2. $(x, y) = (-1, 0)$, $\lambda = \frac{7}{2}$.
3. $(x, y) = \left(-\frac{1}{2}, \sqrt{3}\right)$, $\lambda = 4$.
4. $(x, y) = \left(-\frac{1}{2}, -\sqrt{3}\right)$, $\lambda = 4$.

Evaluate $f(x, y)$ at each point:

1. For $(1, 0)$:

$$f(1, 0) = 3(1)^2 + (0)^2 - 1 + 1 = 3 - 1 + 1 = \mathbf{3}.$$

2. For $(-1, 0)$:

$$f(-1, 0) = 3(-1)^2 + (0)^2 - (-1) + 1 = 3 + 1 + 1 = \mathbf{5}.$$

3. For $\left(-\frac{1}{2}, \sqrt{3}\right)$:

$$\begin{aligned} f\left(-\frac{1}{2}, \sqrt{3}\right) &= 3\left(-\frac{1}{2}\right)^2 + (\sqrt{3})^2 - \left(-\frac{1}{2}\right) + 1 \\ &= 3\left(\frac{1}{4}\right) + 3 + \frac{1}{2} + 1 \\ &= \frac{3}{4} + 3 + \frac{1}{2} + 1 \\ &= \frac{3}{4} + \frac{1}{2} + 4 \\ &= \frac{5}{4} + 4 = \frac{5}{4} + \frac{16}{4} = \frac{21}{4} = \mathbf{5.25}. \end{aligned}$$

4. For $\left(-\frac{1}{2}, -\sqrt{3}\right)$:

$$f\left(-\frac{1}{2}, -\sqrt{3}\right) = \mathbf{5.25} \quad \text{(same value as above)}.$$

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4. $(x, y) = \left(-\frac{1}{2}, -\sqrt{3}\right)$, $\lambda = 4$.

Hessian analysis for each critical point
The bordered Hessian for problems with one constraint is defined as:

$$H = \begin{pmatrix} 0 & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \frac{\partial g}{\partial y} & \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix}$$

We calculate the second derivatives:

$$\frac{\partial^2 \mathcal{L}}{\partial x^2} = 6 - 2\lambda, \quad \frac{\partial^2 \mathcal{L}}{\partial y^2} = 2 - \frac{\lambda}{2}, \quad \frac{\partial^2 \mathcal{L}}{\partial x \partial y} = \frac{\partial^2 \mathcal{L}}{\partial y \partial x} = 0.$$

And the derivatives of g:

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = \frac{y}{2}.$$

We proceed to evaluate the Hessian at each critical point.

Conclusion: The point $(1,0)$ is a local minimum with value $f(1,0) = 3$. The point $(-1,0)$ is a local minimum with value $f(-1,0) = 5$. The points $\left(-\frac{1}{2}, \pm\sqrt{3}\right)$ are local maxima.